7.3 Constrained Optimization

Example (Typical problem)

Consider the quadratic form:

$$\boxed{Q(x_1,x_2,x_3) = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + x_1x_3 + x_2x_3}$$

Find the maximum value of Q Subject to the Constraint $\|\vec{x}\| = 1$.

Answ:

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

Where
$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Since A is Symmetric, it is orthogonally diagonalizable

It Can be easily proved that

Eigenvalues of A: $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 1$

Eigenvectors of A: $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 =$ $\vec{u}_3 =$

Using the ppal axes theorem $Q(\vec{x})$ can be Transformed into a quadratic form with no cross-product terms.

If
$$\vec{x} = P\vec{y}, \qquad \text{where } P = [\hat{u}_1\ \hat{u}_2\ \hat{u}_3]$$

$\hat{u}_1, \hat{u}_2, \hat{u}_3$ are corresponding normalized eigenvectors for $\lambda_1, \lambda_2, \lambda_3$.

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T (P^T A P) \vec{y}$$
$$\overset{A \text{ Symm.}}{=} \vec{y}^T D \vec{y} = Q(\vec{y})$$

where
$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It means Q in the new variables y_1, y_2, y_3 transforms into
$$Q(y_1, y_2, y_3) = 6y_1^2 + 3y_2^2 + y_3^2.$$

Obviously,
$$Q(\vec{y}) = 6y_1^2 + 3y_2^2 + y_3^2 \le 6y_1^2 + 6y_2^2 + 6y_3^2 \le 6(y_1^2 + y_2^2 + y_3^2)$$

If $\|\vec{y}\| = 1 \Rightarrow \boxed{Q(\vec{y}) \le 6}$ for all $\vec{y}$ s.t. $\|\vec{y}\| = 1$.

Also, if $\|\vec{y}\|=1$

$$Q(\vec{y}) = 6y_1^2 + 3y_2^2 + y_3^2 \geq y_1^2 + y_2^2 + y_3^2 = 1$$

So $\boxed{Q(\vec{y}) \geq 1}$, for $\|\vec{y}\|=1$

Since

$Q(1,0,0) = 6$ $\quad \underset{\vec{y}}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \xrightarrow{P} \underset{\vec{x}}{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}}$ Eigenvector of $\lambda_1 = 6$

The maximum is reached when $\vec{y} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Similarly,

minimum $\boxed{Q(0,0,1) = 1}$

Theorem 6 (Book).